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Nonlocal Artificial Boundary Conditions for the Incompressible Viscous Flow in a Channel Using Spectral Techniques

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In this paper the numerical simulation of the steady incompressible viscous flow in a no-slip channel is considered. A sequence of approximate nonlocal artificial boundary conditions on a given segment artificial boundary is derived by a system of linearized Navier–Stokes equations and spectral techniques. Then the original problem is reduced to a boundary value problem in a bounded computational domain. The numerical examples show that these artificial boundary conditions are very effective and are also more accurate than Dirichlet and Neumann boundary conditions, which are often used in the engineering literature. © 1996 Academic Press, Inc.

1. INTRODUCTION

Many problems arising in fluid flow lead to the resolution of a system of partial differential equations in an unbounded domain. For instance, for steady state incompressible viscous flow in a channel, the resolution of Navier–Stokes (N–S) equations in an unbounded domain is proposed. One difficulty in the numerical simulations of these problems is the unboundedness of the physical domain. Various strategies have been developed for overcoming the difficulty [1]. It is a popular method in the engineering literature to introduce an artificial boundary reducing these problems to a bounded computational domain and to set up artificial boundary conditions at the artificial boundary. How to design artificial boundary conditions on the artificial boundary for a given problem has been a common interest for mathematicians and engineers. In the past 10 years, many authors have worked in this direction. For example, Goldstein [2], Feng [3], Han and Wu [4, 5], Hagstrom and Keller [6, 7], Halpern and Schatzman [8, 9], Han *et al.* [10], Han and Bao [11, 12], and Nataf [13] worked on this subject for various problems using different techniques. In [6, 7], the authors proposed a method by which to derive asymptotic boundary conditions for linear partial differential equations in cylinders, which was applied to solve some nonlinear problems.

The purpose of this paper is to design nonlocal artificial boundary conditions for steady incompressible viscous flow in the vorticity streamfunction formulation in the case when the domain is a no-slip channel. We introduce two-segment artificial boundaries in the physical domain. The spectral Chebyshev Tau method [14] is used to design the artificial boundary condition. Then the original problem is reduced to a problem on a bounded computational domain. Finally numerical examples show that the artificial boundary conditions given in this paper are very effective.

2. NAVIER–STOKES EQUATIONS AND THEIR LINEARIZATION

Throughout this paper we consider the numerical simulation of a steady incompressible viscous flow around a body (domain Ω_i) in a no-slip channel defined by $\Re \times [0, L]$. Let u, v denote the components of the velocity in the x and y coordinate directions, and let p denote the pressure; then in the domain $\Omega = \Re \times [0, L]\backslash\overline{\Omega}_i$ u, v, and p satisfy the N–S equations.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \nu\,\Delta u, \tag{2.1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \nu\,\Delta v, \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

and the boundary conditions

$$u|_{y=0,L} = v|_{y=0,L} = 0, \quad -\infty < x < +\infty, \tag{2.4}$$

$$u|_{\partial\Omega_i} = v|_{\partial\Omega_i} = 0, \tag{2.5}$$

$$\begin{aligned} &u(x, y) \to u_\infty(y) = 4ay(L-y)/L^2, \\ &v(x, y) \to v_\infty = 0, \quad \text{when } x \to \pm\infty, \end{aligned} \tag{2.6}$$

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TABLE I

Re = 20, $c = d = 3.0$

Errors	i = I	i = II	i = III
$\mathrm{err}(\omega_E - \omega_i)(d)$	0.1892	0.1125	1.4529×10^{-3}
$\mathrm{err}(\psi_E - \psi_i)(d)$	4.3653×10^{-3}	5.2035×10^{-3}	5.1546×10^{-5}

TABLE II

Re = 50, $c = d = 3.0$

Errors	i = I	i = II	i = III
$\mathrm{err}(\omega_E - \omega_i)(d)$	0.5501	0.2442	1.3080×10^{-2}
$\mathrm{err}(\psi_E - \psi_i)(d)$	2.0178×10^{-2}	9.4361×10^{-3}	4.4893×10^{-4}

where $\nu > 0$ is the kinematic viscosity, and $a > 0$ is a constant.

We introduce the streamfunction ψ and vorticity ω; then

$$\frac{\partial\psi}{\partial y} = u, \quad \frac{\partial\psi}{\partial x} = -v, \tag{2.7}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2.8}$$

Thus the problem (2.1)–(2.6) is equivalent to the problem

$$\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \nu\Delta\omega = 0, \quad \text{in } \Omega, \tag{2.9}$$

$$\Delta\psi + \omega = 0, \quad \text{in } \Omega, \tag{2.10}$$

$$\psi|_{y=0} = \frac{\partial\psi}{\partial y}|_{y=0,L} = 0, \quad \psi|_{y=L} = \psi_L$$
$$\equiv \int_0^L u_\infty(s)\,ds, \quad -\infty < x < +\infty, \tag{2.11}$$

$$\psi|_{\partial\Omega_i} = \text{constant}, \quad \frac{\partial\psi}{\partial n}|_{\partial\Omega_i} = 0, \tag{2.12}$$

$$\psi(x, y) \to \psi_\infty(y) \equiv \int_0^y u_\infty(s)\,ds,$$
$$\omega(x, y) \to \omega_\infty(y) \equiv -u'_\infty(y), \text{ when } x \to \pm\infty. \tag{2.13}$$

We take two constants $b < c$, such that $\overline{\Omega}_i \subset (b, c) \times (0, L)$; then Ω is divided into three parts, Ω_b, Ω_T, and Ω_c, by the artifical boundaries Γ_b and Γ_c with

$$\Gamma_b = \{(x, y) \mid x = b, 0 \le y \le L\},$$
$$\Gamma_c = \{(x, y) \mid x = c, 0 \le y \le L\},$$
$$\Omega_b = \{(x, y) \mid -\infty < x < b, 0 < y < L\},$$
$$\Omega_T = \{(x, y) \mid b < x < c, 0 < y < L\} \backslash \overline{\Omega}_i,$$
$$\Omega_c = \{(x, y) \mid c < x < +\infty, 0 < y < L\}.$$

When $|b|$, c are sufficiently large, in the domain $\Omega_b \cup \Omega_c$ the flow is almost a Poiseuille flow. So the nonlinear N–S equations (2.9), (2.10) can be linearized; namely, in the domain Ω_c (and Ω_b) the solution ω and ψ of the problem (2.9)–(2.13) approximately satisfies the problem

$$\Delta\omega - u_\infty(y)\mathrm{Re}\frac{\partial\omega}{\partial x} - u''_\infty(y)\mathrm{Re}\frac{\partial\psi}{\partial x} = 0, \quad \text{in } \Omega_c, \tag{2.14}$$

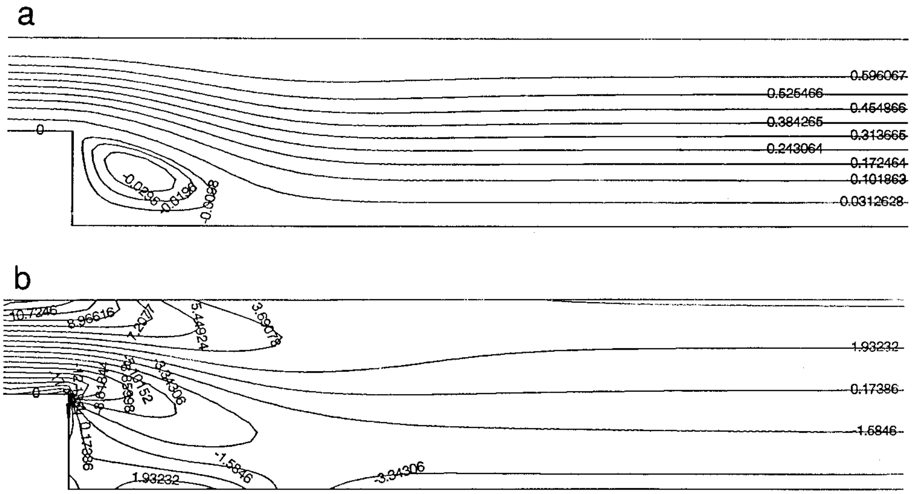


FIG. 1. (a) Streamfunction, exact solution; (b) vorticity, exact solution.

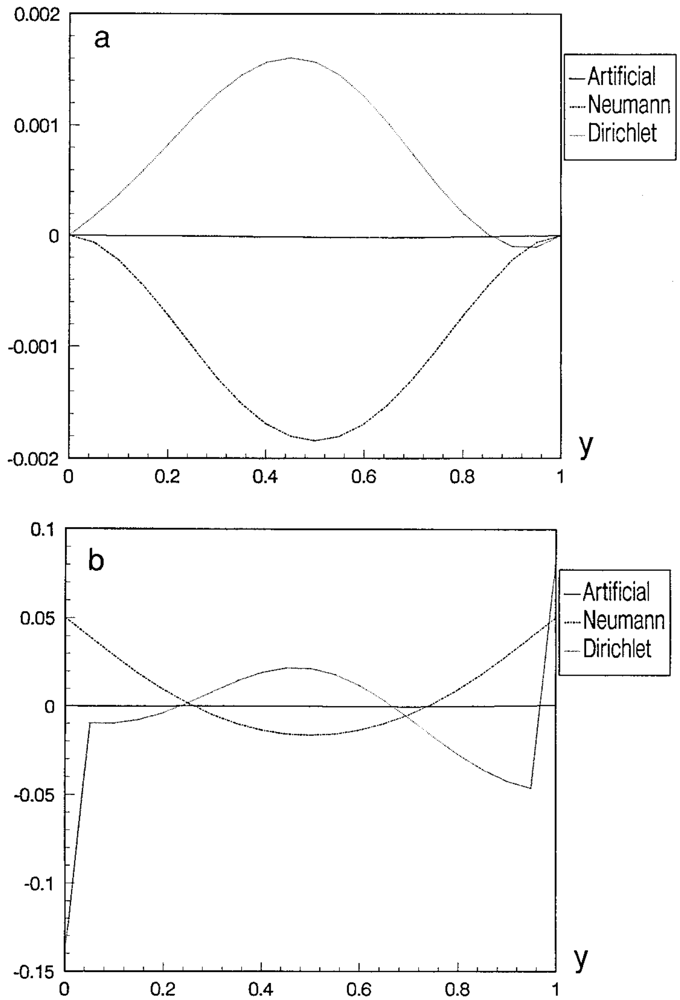


FIG. 2. Re = 20, $c = d = 3.0$: (a) $\psi_E - \psi_i$; (b) $\omega_E - \omega_i$.

$$\Delta\psi + \omega = 0, \quad \text{in } \Omega_c, \tag{2.15}$$

$$\psi|_{y=0} = \frac{\partial\psi}{\partial y}|_{y=0,L} = 0, \quad \psi|_{y=L} = \psi_L, \quad c \le x < +\infty, \tag{2.16}$$

$$\psi(x,y) \to \psi_\infty(y), \quad \omega(x,y) \to \omega_\infty(y), \quad \text{when } x \to +\infty, \tag{2.17}$$

where Re $= 1/\nu$. Let

TABLE III

Re = 100, $c = d = 3.0$

Errors	i = I	i = II	i = III
$\mathrm{err}(\omega_E - \omega_i)(d)$	2.3946	0.3422	5.2694×10^{-2}
$\mathrm{err}(\psi_E - \psi_i)(d)$	5.5058×10^{-2}	7.0047×10^{-3}	1.8079×10^{-3}

$$\tilde{\omega}(x,y) = \omega(x,y) - \omega_\infty(y),$$

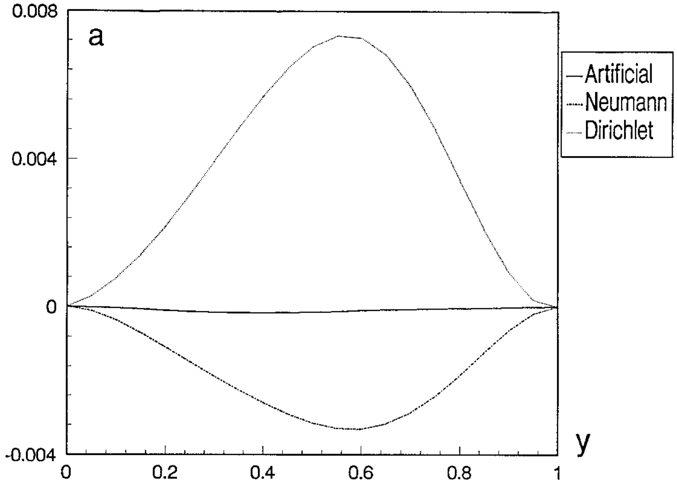
$$\tilde{\psi}(x,y) = \psi(x,y) - \psi_\infty(y).$$

Since $\psi_\infty(y)$ is a polynomial of degree three, $\omega_\infty(y)$ is a polynomial of degree one and $\psi''_\infty(y) + \omega_\infty(y) = 0$, so it is straightforward to check that $\tilde{\omega}$ and $\tilde{\psi}$ satisfy the equations (2.14), (2.15) and the boundary conditions

$$\tilde{\psi}|_{y=0,L} = \frac{\partial\tilde{\psi}}{\partial y}|_{y=0,L} = 0, \quad c \le x < +\infty, \tag{2.18}$$

$$\tilde{\psi}(x,y) \to 0, \quad \tilde{\omega}(x,y) \to 0, \quad \text{when } x \to +\infty. \tag{2.19}$$

Since the boundary condition on the artificial boundary Γ_c is unknown, the equations (2.14) and (2.15) with boundary conditions (2.18) and (2.19) are an incompletely posed problem. It cannot be solved. Let



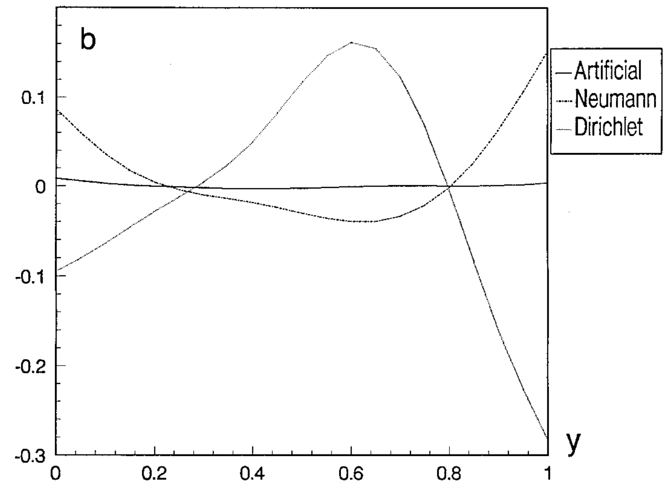


FIG. 3. Re = 50, $c = d = 3.0$: (a) $\psi_E - \psi_i$; (b) $\omega_E - \omega_i$.

$$\tilde{\psi}(x, y)|_{x=c} = \tilde{\psi}_c(y), \quad \tilde{\omega}(x, y)|_{x=c} = \tilde{\omega}_c(y), \quad 0 \le y \le L. \tag{2.20}$$

For given functions $\tilde{\psi}_c(y)$ and $\tilde{\omega}_c(y)$ with $\tilde{\psi}_c(0) = \tilde{\psi}_c(L) = 0$ and $(d\tilde{\psi}_c(y)/dy)|_{y=0,L} = 0$, we discuss the solution of the equations (2.14) and (2.15) with the boundary conditions (2.18)–(2.20) and design a sequence of artificial boundary conditions on the segment Γ_c for the problem (2.9)–(2.13).

3. ARTIFICIAL BOUNDARY CONDITIONS

We now solve the equations (2.14) and (2.15) with the boundary conditions (2.18)–(2.20) by spectral techniques and then design a sequence of approximate artificial boundary conditions on the segment Γ_c for the problem (2.9)–(2.13). Suppose

$$\tilde{\omega}(x, y) = \alpha(y)e^{2\lambda(x-c)/L}$$

$$\tilde{\psi}(x, y) = \beta(y)e^{2\lambda(x-c)/L}$$

is a nonzero solution of the problem (2.14), (2.15), (2.18), (2.19). Then we know that the constant λ and the nonzero functions $\alpha(y)$ and $\beta(y)$ are a solution of the eigenvalue problem

$$\frac{4}{L^2}\lambda^2\alpha(y) + \alpha''(y) - \frac{2\lambda\mathrm{Re}}{L}u_\infty(y)\alpha(y) - \frac{2\lambda\mathrm{Re}}{L}u''_\infty(y)\beta(y) = 0, \tag{3.1}$$

$$\frac{4}{L^2}\lambda^2\beta(y) + \beta''(y) + \alpha(y) = 0, \quad 0 < y < L, \tag{3.2}$$

$$\beta(y)|_{y=0,L} = \beta'(y)|_{y=0,L} = 0, \tag{3.3}$$

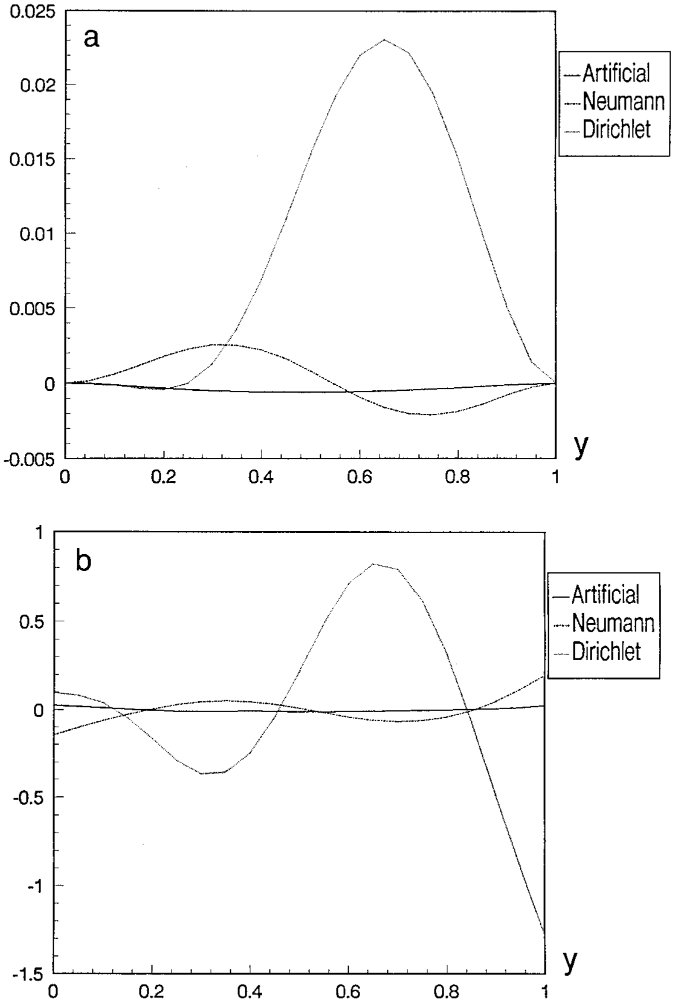


FIG. 4. Re = 100, $c = d = 3.0$: (a) $\psi_E - \psi_i$; (b) $\omega_E - \omega_i$.

$$\text{Real}\lambda < 0, \tag{3.4}$$

where Realλ denotes the real part of λ. Let $y = L(t+1)/2$ and

$$\tilde{\alpha}(t) = \frac{L^2}{4}\alpha(y) = \frac{L^2}{4}\alpha\left(\frac{L(t+1)}{2}\right),$$

$$\tilde{\beta}(t) = \beta(y) = \beta\left(\frac{L(t+1)}{2}\right), \quad -1 \le t \le 1, 0 \le y \le L,$$

$$\tilde{u}_\infty(t) = u_\infty(y) = u_\infty\left(\frac{L(t+1)}{2}\right) = a(1-t^2).$$

Then we obtain

$$\lambda^2\tilde{\alpha}(t) + \tilde{\alpha}''(t) - \frac{L\lambda\text{Re}}{2}\tilde{u}_\infty(t)\tilde{\alpha}(t) - \frac{L\lambda\text{Re}}{2}\tilde{u}''_\infty(t)\tilde{\beta}(t) = 0, \tag{3.5}$$

$$\lambda^2\tilde{\beta}(t) + \tilde{\beta}''(t) + \tilde{\alpha}(t) = 0, \quad -1 < t < 1, \tag{3.6}$$

$$\tilde{\beta}(t)|_{t=\pm 1} = \tilde{\beta}'(t)|_{t=\pm 1} = 0, \tag{3.7}$$

$$\text{Real}\lambda < 0. \tag{3.8}$$

In the following we solve the problem (3.5)–(3.8) by the spectral Chebyshev Tau method [14]. First we give an equivalent form of the boundary condition (3.7). We have the following theorem.

THEOREM. The boundary conditions $\tilde{\beta}(t)|_{t=\pm 1} = \tilde{\beta}'(t)|_{t=\pm 1} = 0$ are equivalent to $\tilde{\beta}(t)|_{t=\pm 1} = 0$ and $\int_{-1}^{1} \tilde{\alpha}\eta \, dt$

TABLE IV

Re = 20, $d = 2.5$

c	2.5	3.0	3.5
$\mathrm{err}(\omega_E - \omega_{III})(d)$	9.0981×10^{-3}	1.5865×10^{-4}	2.0587×10^{-5}
$\mathrm{err}(\psi_E - \psi_{III})(d)$	2.0359×10^{-4}	6.8568×10^{-6}	9.3753×10^{-7}

TABLE VI

Re = 100, $d = 2.5$

c	2.5	3.0	3.5
$\mathrm{err}(\omega_E - \omega_{III})(d)$	0.5073	2.0992×10^{-2}	1.3615×10^{-2}
$\mathrm{err}(\psi_E - \psi_{III})(d)$	2.3621×10^{-2}	8.8396×10^{-4}	5.7431×10^{-4}

$= -\lambda^2 \int_{-1}^{1} \tilde{\beta}\eta\, dt$ for any $\eta \in P_1(-1, 1)$ if $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$ satisfy the equation $\lambda^2\tilde{\beta}(t) + \tilde{\beta}''(t) + \tilde{\alpha}(t) = 0$ on $(-1, 1)$.

Proof. For any function $\eta(t) \in P_1(-1, 1)$,

$$\begin{aligned}\int_{-1}^{1} \tilde{\alpha}(t)\eta(t)\, dt &= -\int_{-1}^{1} [\lambda^2\tilde{\beta}(t) + \tilde{\beta}''(t)]\,\eta(t)\, dt \\ &= -\lambda^2 \int_{-1}^{1} \tilde{\beta}\eta\, dt - \int_{-1}^{1} \tilde{\beta}''\eta\, dt \\ &= -\lambda^2 \int_{-1}^{1} \tilde{\beta}\eta\, dt - \int_{-1}^{1} [\tilde{\beta}''\eta - \eta''\tilde{\beta}]\, dt \\ &= -\lambda^2 \int_{-1}^{1} \tilde{\beta}\eta\, dt - [\tilde{\beta}'\eta - \eta'\tilde{\beta}]|_{-1}^{1},\end{aligned}$$

since $[\tilde{\beta}\eta' - \eta\tilde{\beta}']|_{-1}^{1} = 0$ for any $\eta \in P_1(1, -1)$ if and only if $\tilde{\beta}(1) - \tilde{\beta}(-1) - \tilde{\beta}'(1) - \tilde{\beta}'(-1) = 0$ and $\tilde{\beta}'(1) - \tilde{\beta}'(-1) = 0$. When this is combined with $\tilde{\beta}(t)|_{t=\pm 1} = 0$, the theorem is completed.

Therefore the boundary condition (3.7) can by replaced by

$$\tilde{\beta}(t)|_{t=\pm 1} = 0, \tag{3.9}$$

$$\begin{aligned}&\int_{-1}^{1} \tilde{\alpha}(t)\, dt = -\lambda^2 \int_{-1}^{1} \tilde{\beta}(t)\, dt \quad \text{and} \\ &\int_{-1}^{1} t\tilde{\alpha}(t)\, dt = -\lambda^2 \int_{-1}^{1} t\tilde{\beta}(t)\, dt.\end{aligned} \tag{3.10}$$

Now we discretize the problem (3.5), (3.6), (3.9), (3.10) by the spectral Chebyshev Tau method [14]. Let

$$\tilde{\alpha}_N(t) = \sum_{j=0}^{n} a_j T_j(t),$$

$$\tilde{\beta}_N(t) = \sum_{j=0}^{N} b_j T_j(T),$$

TABLE V

Re = 50, $d = 2.5$

c	2.5	3.0	3.5
$\mathrm{err}(\omega_E - \omega_{III})(d)$	1.4283×10^{-2}	4.8064×10^{-3}	1.6527×10^{-3}
$\mathrm{err}(\psi_E - \psi_{III})(d)$	2.2518×10^{-4}	2.0739×10^{-4}	6.9529×10^{-5}

where $T_j(t)$ is the jth Chebyshev polynomial and N is even. A computation shows that

$$\begin{aligned}&\lambda^2 a_k + \frac{1}{c_k} \sum_{\substack{p=k+2 \\ p+k \text{ even}}}^{N} p(p^2 - k^2)a_p - \frac{aL\lambda\mathrm{Re}}{2} \sum_{p=0}^{N} \gamma_{pk} a_p \\ &\qquad + aL\mathrm{Re}\lambda b_k = 0, \\ &k = 0, 1, ..., N-2,\end{aligned} \tag{3.11}$$

$$\lambda^2 b_k + \frac{1}{c_k} \sum_{\substack{p=k+2 \\ p+k \text{ even}}}^{N} p(p^2 - k^2)b_p + a_k = 0, k = 0, 1, ..., N-2, \tag{3.12}$$

and

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{N} b_j = \sum_{\substack{j=1 \\ j \text{ odd}}}^{N} b_j = 0, \tag{3.13}$$

$$\begin{aligned}&\sum_{\substack{j=0 \\ j \text{ even}}}^{N} \frac{a_j}{j^2 - 1} = -\lambda^2 \sum_{\substack{j=0 \\ j \text{ even}}}^{N} \frac{b_j}{j^2 - 1} \quad \text{and} \\ &\sum_{\substack{j=1 \\ j \text{ odd}}}^{N} \frac{a_j}{j^2 - 4} = -\lambda^2 \sum_{\substack{j=1 \\ j \text{ odd}}}^{N} \frac{b_j}{j^2 - 4},\end{aligned} \tag{3.14}$$

where

$$c_k = \begin{cases} 2, & k = 0, \\ 1, & k = 1, 2, ..., N, \end{cases}$$

$$\gamma_{jk} = \begin{cases} \frac{1}{4}[c_k\delta_{j,k} - \frac{1}{2}\delta_{2,j+k} - \frac{1}{2}\delta_{2,|j-k|}], & 0 \le j \le N-1, 0 \le k \le N, \\ 0, & j = N-1, N, 0 \le k \le N, \end{cases}$$

and

$$\delta_{j,k} = \begin{cases} 1, & j = k, \\ 0, & j \ne k. \end{cases}$$

Let

$$X = [a_0, a_1, ..., a_{N-2}, b_0, b_1, ..., b_{N-2}]^{\mathrm{T}}.$$

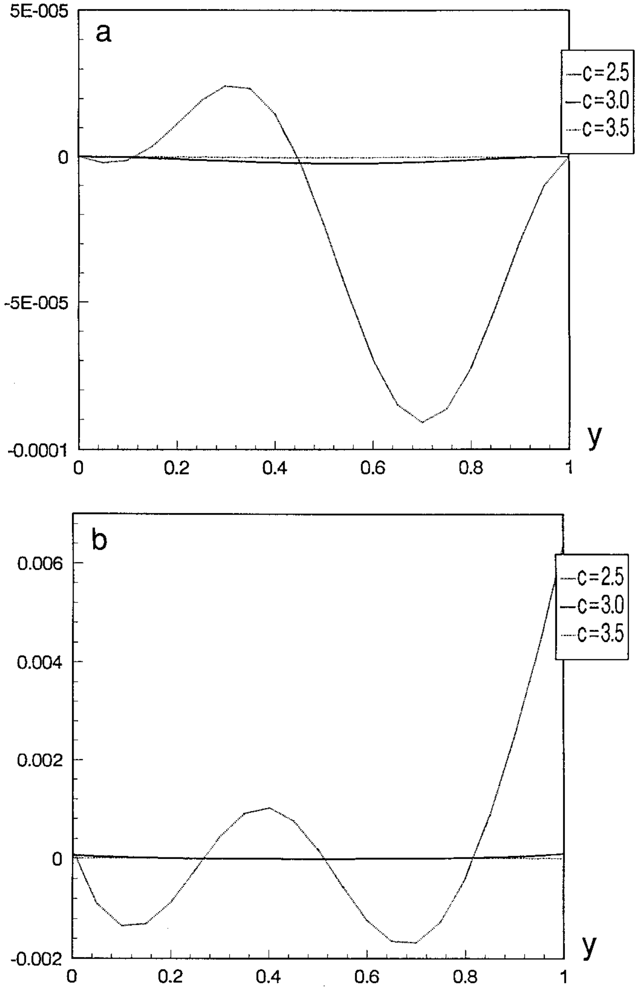


FIG. 5. Re = 20, $d = 2.5$: (a) $\psi_E - \psi_{III}$; (b) $\omega_E - \omega_{iii}$.

Then the problem (3.11)–(3.14) is equivalent to the eigenvalue problem

$$\lambda^2 X + \lambda A_0 X + B_0 X = 0, \tag{3.15}$$

whereA_0 and B_0 are $(2N - 2) \times (2N - 2)$ matrices. Let

$$Y = \lambda X.$$

Then (3.15) is reduced to the standard eigenvalue problem

$$\begin{pmatrix} 0 & I_{2N-2} \\ -B_0 & -A_0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{3.16}$$

where I_{2N-2} is a $(2N - 2) \times (2N - 2)$ unit matrix.

From the condition (3.4), the real part of λ must be negative, so to solve the problem (3.1)–(3.4), we need only to calculate the eigenvalues with negative real part and the corresponding eigenvectors of the eigenvalue problem (3.16). We can compute all the eigenvalues with negative real part by numerical methods. Therefore we assume that the eigenvalues of problem (3.16) with negative real part are $\lambda_1, \lambda_2, ..., \lambda_K$, and the corresponding eigenvectors are $\xi_1, \xi_2, ..., \xi_K$. Furthermore we suppose that $\mathrm{Real}\lambda_i \geq \mathrm{Real}\lambda_{i+1}$ $(1 \leq i \leq K - 1)$ and $\xi_i = (\xi_{1,i}, \xi_{2,i}, ..., \xi_{2N-2,i})^T$ $(1 \leq i \leq K)$. We now design a sequence of approximate artificial boundary conditions on the segment Γ_c using $\lambda_1, \lambda_2, ..., \lambda_K$ and $\xi_1, \xi_2, ..., \xi_K$. We introduce $\eta_i^\omega = (\eta_{1,i}^\omega, ..., \eta_{N+1,i}^\omega)^T$ and $\eta_i^\psi = (\eta_{1,i}^\psi, ..., \eta_{N+1,i}^\psi)^T \in \Re^{N+1}$ $(1 \leq i \leq K)$, with

$$\eta_{j,i}^\psi = \begin{cases} \xi_{N-1+j,i}, & 1 \leq j \leq N-1, \\ -\sum\limits_{\substack{l=1 \\ l \text{ odd}}}^{N-2} \xi_{N-1+l,i}, & j = N, \\ -\sum\limits_{\substack{l=0 \\ l \text{ even}}}^{N-2} \xi_{N-1+l,i}, & j = N+1, \end{cases}$$

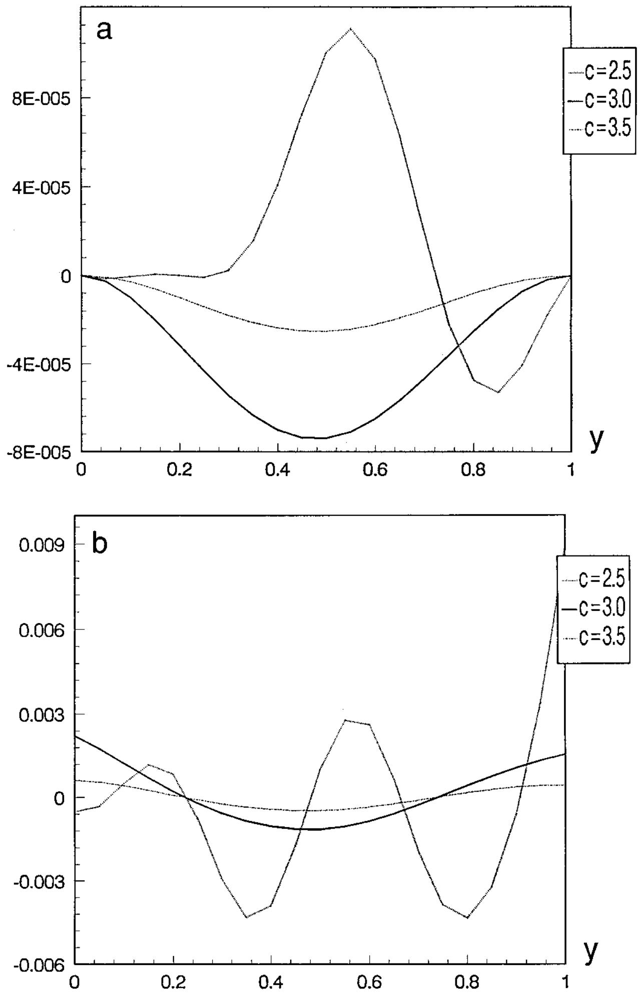


FIG. 6. Re = 50, d = 2.5: (a) $\psi_{\mathrm{E}} - \psi_{\mathrm{III}}$; (b) $\omega_{\mathrm{E}} - \omega_{\mathrm{iii}}$.

$$\eta^{\omega}_{j,i} = \begin{cases} \xi_{j,i}, & 1 \le j \le N-1, \\ -\displaystyle\sum_{\substack{l=1 \\ l\,\text{odd}}}^{N-2} \left[\frac{(N-1)^2 - 4}{l^2 - 4}\, \xi_{l,i} + \frac{(N-1)^2 - l^2}{l^2 - 4}\, \lambda_i^2\, \xi_{N-1+l,i} \right], & j = N, \\ -\displaystyle\sum_{\substack{l=0 \\ l\,\text{even}}}^{N-2} \left[\frac{N^2 - 1}{l^2 - 1}\, \xi_{l,i} + \frac{N^2 - l^2}{l^2 - 1}\, \lambda_i^2\, \xi_{N-1+l,i} \right], & j = N+1. \end{cases}$$

Thus $\alpha_i(t) = \sum_{j=0}^{N} \eta^{\omega}_{j+1,i} T_j(t)$ and $\beta_i(t) = \sum_{j=0}^{N} \eta^{\psi}_{j+1,i} T_j(t)$ are approximate eigenfunctions of the eigenvalues λ_i of the problem (3.5)–(3.8). Suppose $M \in \aleph$ and $M \le K$. Let

$$\tilde{\omega}_M(x, y) = \sum_{i=1}^{M} d_i g_i^{\omega}(x, y), \tag{3.17}$$

$$\tilde{\psi}_M(x, y) = \sum_{i=1}^{M} d_i g_i^{\psi}(x, y), \tag{3.18}$$

where

$$g_i^{\omega}(x, y) = \frac{4}{L^2} \begin{cases} e^{2\lambda_i (x-c)/L}\, \alpha_i\left(\dfrac{2y}{L} - 1\right), & \lambda_i \text{ real}, \\ \mathrm{Real}\left[e^{2\lambda_i (x-c)/L}\, \alpha_i\left(\dfrac{2y}{L} - 1\right) \right], & \mathrm{Imag}\lambda_i \ne 0,\ \lambda_i = \overline{\lambda}_{i+1}, \\ \mathrm{Imag}\left[e^{2\lambda_i (x-c)/L}\, \alpha_i\left(\dfrac{2y}{L} - 1\right) \right], & \mathrm{Imag}\lambda_i \ne 0,\ \lambda_i \ne \overline{\lambda}_{i+1} \end{cases}$$

and

$$g_i^{\psi}(x,y) = \begin{cases} e^{2\lambda_i(x-c)/L}\,\beta_i\left(\dfrac{2y}{L}-1\right), & \lambda_i \text{ real},\\ \text{Real}\left[e^{2\lambda_i(x-c)/L}\,\beta_i\left(\dfrac{2y}{L}-1\right)\right], & \text{Imag}\lambda_i \neq 0,\ \lambda_i = \overline{\lambda}_{i+1},\\ \text{Imag}\left[e^{2\lambda_i(x-c/L}\,\beta_i\left(\dfrac{2y}{L}-1\right)\right], & \text{Imag}\lambda_i \neq 0,\ \lambda_i \neq \overline{\lambda}_{i+1}, \end{cases}$$

where Imag(s) denotes the imaginary part of s and $\overline{\lambda}$ denotes conjugation of λ.

Therefore $\tilde{\omega}_M(x, y)$ and $\tilde{\psi}_M(x, y)$ satisfy the equations (2.14)–(2.15) and boundary conditions (2.18)–(2.19) approximately for any constants $d_1, d_2, ..., d_M$. We introduce

$$g_i(y) = \begin{pmatrix} g_i^{\omega}(c,y) \\ g_i^{\psi}(c,y) \end{pmatrix}, 1 \le i \le M, g_\infty(y) = \begin{pmatrix} \omega_\infty(y) \\ \psi_\infty(y) \end{pmatrix},$$

$$W_M(x,y) = \begin{pmatrix} \omega_M(x,y) \\ \psi_M(x,y) \end{pmatrix}, \tilde{W}_M(x,y) = \begin{pmatrix} \tilde{\omega}_M(x,y) \\ \tilde{\psi}_M(x,y) \end{pmatrix}.$$

Hence we have

$$\tilde{W}_M(c,y) = \sum_{i=1}^{M} d_i g_i(y), \tag{3.19}$$

$$W_M(c,y) = \tilde{W}_M(c,y) + g_\infty(y). \tag{3.20}$$

Suppose $w(y) = 1/\sqrt{1-(2y/L-1)^2}$ is the Chebyshev weight function on the interval $[0, L]$. Then we obtain

$$d = D^{-1}r, \tag{3.21}$$

where

$$\begin{aligned} d &= [d_1, d_2, ..., d_M]^{\mathrm{T}},\\ r &= \left[\int_0^L w(y)\tilde{W}_M(c,y)\cdot g_1(y)\,dy, ..., \int_0^L w(y)\tilde{W}_M(c,y)\cdot g_M(y)\,dy\right]^{\mathrm{T}}\\ &= \left[\int_0^L w(y)W_M(c,y)\cdot g_1(y)\,dy, ..., \int_0^L w(y)W_M(c,y)\cdot g_M(y)\,dy\right]^{\mathrm{T}}\\ &\quad - \left[\int_0^L w(y)g_\infty(y)\cdot g_1(y)\,dy, ..., \int_0^L w(y)g_\infty(y)\cdot g_M(y)\,dy\right]^{\mathrm{T}}, \end{aligned}$$

and

$$D = \begin{pmatrix} \int_0^L w(y)g_1(y)\cdot g_1(y)\,dy & \dots & \int_0^L w(y)g_M(y)\cdot g_1(y)\,dy \\ \dots & \ddots & \dots \\ \int_0^L w(y)g_1(y)g_M(y)\cdot dy & \dots & \int_0^L w(y)g_M(y)\cdot g_M(y)\,dy \end{pmatrix}.$$

Differentiating (3.17) and (3.18) and let $x = c$, we obtain

$$\begin{aligned} \frac{\partial W_M(c,y)}{\partial x} &= \frac{\partial \tilde{W}_M(c,y)}{\partial x}\\ &= \sum_{i=1}^{M} h_i(y)\,d_i\\ &= (h_1(y), ..., h_M(y))D^{-1}r\\ &\equiv \prod_M^c(\omega,\psi) \quad 1 \le M \le K, \end{aligned} \tag{3.22}$$

where

$$h_i(y) = \begin{pmatrix} \dfrac{\partial g_i^{\omega}(c,y)}{\partial x} \\ \dfrac{\partial g_i^{\psi}(c,y)}{\partial x} \end{pmatrix}, \quad 1 \le i \le M.$$

Therefore we obtain a sequence of approximate artificial boundary conditions (3.22) on the segment artificial boundary Γ_c.

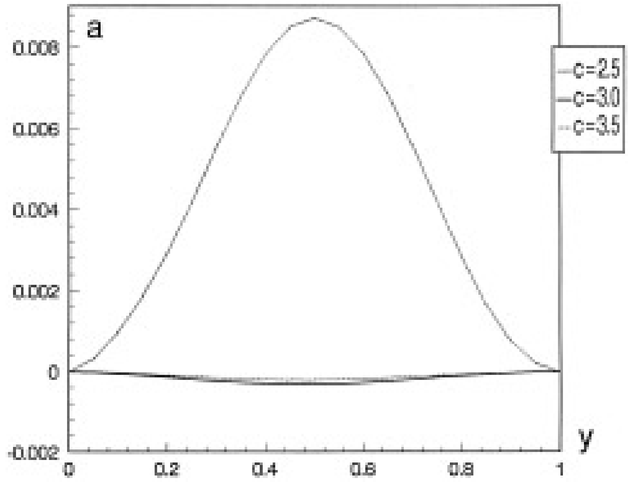
In a similar way, we can get the artificial boundary conditions on the boundary Γ_b.

Then on the domain Ω_T the original problem (2.9)–(2.13) can be approximated by the following problem with different M:

$$\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \nu\,\Delta\omega = 0, \quad \text{in } \Omega_T, \tag{3.23}$$

$$\Delta\psi + \omega = 0, \quad \text{in } \Omega_T, \tag{3.24}$$

$$\psi\Big|_{y=0} = \frac{\partial\psi}{\partial y}\Big|_{y=0,L} = 0, \quad \psi|_{y=L} = \psi_L, \quad b \le x \le c, \tag{3.25}$$



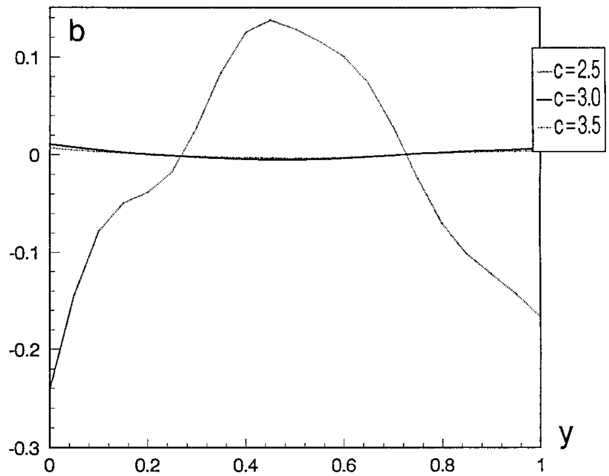


FIG. 7. Re = 100, d = 2.5: (a) $\psi_{\mathrm{E}} - \psi_{\mathrm{III}}$; (b) $\omega_{\mathrm{E}} - \omega_{\mathrm{iii}}$.

$$\psi|_{\partial\Omega_i} = \text{constant}, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega_i} = 0, \tag{3.26}$$

$$\psi|_{\Gamma_b} = \psi_\infty(y), \qquad \omega|_{\Gamma_b} = \omega_\infty(y), \tag{3.27}$$

$$\left.\begin{pmatrix} \dfrac{\partial\omega}{\partial x} \\ \dfrac{\partial\psi}{\partial x} \end{pmatrix}\right|_{\Gamma_c} = \prod_M^c (\omega, \psi). \tag{3.28}$$

4. NUMERICAL IMPLEMENTATION AND RESULTS

In this section we consider the numerical solution of the original problem (2.9)–(2.13) on the given computational domain Ω_T. This steady state solution is computed as the limit in time of the unsteady N–S equations, which are discretized by an ADI method [15].

EXAMPLE [Backward-Facing Step Flow]. The bounded computational domain is given by

$$\Omega_T = \left\{(x, y) \mid b < x \le b + \frac{L}{2}, \frac{L}{2} < y < L; \right.$$
$$\left. b + \frac{L}{2} < x < c, 0 < y < L \right\}.$$

Then the inflow condition at the boundary $\Gamma_b = \{(x, y) \mid x = b, L/2 \le y \le L\}$ is given by

$$\omega(b, y) = \frac{16a}{L^2}(4y - 3L),$$

$$\psi(b, y) = \frac{8a}{3L^2}\left(y - \frac{L}{2}\right)^2 (5L - 4y).$$

Thus $\psi_\infty(y)$ and $\omega_\infty(y)$ are given by

TABLE VII

Re = 20, $c = d = 3.0$

M	0	1	2
$\text{err}(\omega_E - \omega_{III})(d)$	0.1125	1.0865×10^{-2}	1.4529×10^{-3}
$\text{err}(\psi_E - \psi_{III})(d)$	5.2035×10^{-3}	5.6575×10^{-4}	5.1546×10^{-5}

TABLE IX

Re = 100, $c = d = 3.0$

M	2	4	6
$\text{err}(\omega_E - \omega_{III})(d)$	0.3643	6.5859×10^{-2}	5.2694×10^{-2}
$\text{err}(\psi_E - \psi_{III})(d)$	1.8544×10^{-2}	2.8569×10^{-3}	1.8079×10^{-3}

$$\omega_\infty(y) = \frac{4a}{L^2}(2y - L),$$

$$\psi_\infty(y) = \frac{4a}{L^2} y^2 \left(\frac{L}{2} - \frac{y}{3}\right).$$

we take $b = 0.0$, $a = 1.0$, $L = 1.0$.

To test the artificial boundary conditions, we made three types of computation using different types of outflow boundary conditions at artificial boundary Γ_c in the example.

Type I. Dirichlet boundary condition

$$\psi(c, y) = \psi_\infty(y), \omega(c, y) = \omega_\infty(y), \quad 0 \le y \le L;$$

Type II. Neumann boundary condition

$$\frac{\partial \psi}{\partial x}(c, y) = 0, \frac{\partial \omega}{\partial x}(c, y) = 0, \quad 0 \le y \le L;$$

Type III. Artificial boundary condition (3.22).

In the example, the results are compared with an "exact solution." This solution is obtained by using an outflow boundary very far from the step, at which are presented Neumann boundary condition. To be precise, the distance between the step and the outflow boundary for the "exact solution" is 14 times the height of the step.

Let (ψ_E, ω_E) denote the "exact solution" and (ψ_i, ω_i) (i = I, II, III) denote the numerical solutions corresponding the boundary conditions type I, II, and III on the artificial boundary Γ_c. Figure 1 shows the "exact solution" for Re = 100. The error $\omega_E(d, y) - \omega_i(d, y)$, $\psi_E(d, y) - \psi_i(d, y)$ on the segment $\Gamma_d = \{(x, y) \mid x = d, 0 \le y \le L\}$ is given. Let

$$\text{err}(f_E - \tilde{f}_i)(d) = \sqrt{\sum_{j=0}^{J} [f_E(d, y_j) - \tilde{f}_i(d, y_i)]^2}.$$

Then the errors $\text{err}(\omega_E - \omega_i)(d)$ and $\text{err}(\psi_E - \psi_i)(d)$ are given in Tables I–III for Re = 20, 50, and 100. Furthermore the errors $\omega_E - \omega_i$ and $\psi_E - \psi_i$ on the segment Γ_d are shown in Figs. 2–4.

Tables I–III and Figs. 2–4 show the artificial boundary condition presented in this paper to be more accurate than the Neumann and Dirichlet boundary conditions, which are often used in the engineering literature.

The influence of the location of the artificial boundary Γ_c is shown in Tables IV–VI and in Figs. 5–7 for different Reynolds numbers. The location of the artificial boundary has strong influence for the computational accuracy, specially for high Reynolds number. The influence of the number M in the artificial boundary condition (3.22) is shown in Tables VII–IX for different Reynolds numbers.

TABLE VIII

Re = 50, $c = d = 3.0$

M	0	2	4
$\text{err}(\omega_E - \omega_{III})(d)$	0.2442	7.8011×10^{-2}	1.3080×10^{-2}
$\text{err}(\psi_E - \psi_{III})(d)$	9.4361×10^{-3}	1.9261×10^{-3}	4.4893×10^{-4}

5. CONCLUSIONS

A sequence of approximate artificial boundary conditions for nonlinear Navier–Stokes equations has been designed using an external linear flow and the spectral Chebyshev Tau method. The artificial boundary conditions can be used to solve nonlinear N–S equations even though it is obtained from linearized N–S equations on an external domain. From the numerical results, we can see that our artificial boundary condition is more accurate than the Neumann and Dirichlet boundary conditions which are often used in the engineering literature. For a given accuracy it is possible to compute the problem on a smaller computational domain using our artificial boundary condition; thus it saves computing time. The numerical results show that the location of the artificial boundary depends on the Reynolds number Re.

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